ERROR ANALYSIS FOR SEMI-ANALYTIC DISPLACEMENT DERIVATIVES WITH RESPECT TO SHAPE AND SIZING VARIABLES

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INTRODUCTION

Sensitivity analysis is fundamental to the solution of structural optimization problems. Consequently, much research has focused on the efficient computation of static displacement derivatives (Ref. 1). As originally developed, these methods relied on analytical representations for the derivatives of the structural stiffness matrix (K) with respect to the design variables (b_i). To extend these methods for use with complex finite element formulations and facilitate their implementation into structural optimization programs (eg. Ref. 2) using general finite element analysis codes (Refs. 3-4), the semi-analytic method (Refs. 5-6) was developed. In this method we approximate the matrix $\partial K / \partial b_i$ by finite difference.

Although it is well known that the accuracy of the semi-analytic method is dependent on the finite difference parameter, recent work (Ref. 7) has suggested that more fundamental inaccuracies exist in the method when used for shape optimization. Another study (Ref. 8) has argued qualitatively (for the case of a cantilevered beam) that these errors are related to non-uniform errors in the stiffness matrix derivatives.

In the following we will investigate the accuracy of the semi-analytic method. We first develop a general framework for the error analysis and then show analytically that the errors in the method are entirely accounted for by errors in $\Delta K / \Delta b_i$. Furthermore, we demonstrate that acceptable accuracy in the derivatives can be obtained through careful selection of the finite difference parameter.

Static displacement derivatives:

$$\frac{\Delta u}{\Delta b_i} = -K^{-1}\frac{\partial K}{\partial b_i}u = -K^{-1}p^s_i$$

In the semi-analytic method we approximate $\dfrac{\Delta K}{\Delta b_i} \cong \dfrac{\partial K}{\partial b_i}$ by finite differences

to compute the approximation $\dfrac{\Delta u}{\Delta b_i} \cong \dfrac{\partial u}{\partial b_i}$ efficiently

This gives the semi-analytic formula $\boxed{\dfrac{\Delta u}{\Delta b_i} = -K^{-1}\dfrac{\Delta K}{\Delta b_i}u = -K^{-1}\bar{p}^s_i}$ which has

been used successfully for sizing optimization.

> Barthelemy & Haftka - demonstrated large errors for shape optimization
> Pedersen, Cheng, & Rasmussen - some analysis of these errors

Figure 1

ACCURACY OF THE SEMI-ANALYTIC METHOD

To characterize the errors associated with the semi-analytic method let us examine the expression for $\Delta u/\Delta b_i$ in terms of the approximate pseudo-load vector $\tilde{p}_i^s$ (see figure 2) and consider two cases. In both cases K is separated into two parts (K and $K(b_i)$) which are independent of and dependent on the design variable b_i, respectively. In the first case we can factor $K(b_i)$ into a constant matrix K_{b_i} and a scalar function $f(b_i)$. As a result, the approximate pseudo-load vector is a simple scaling of the true pseudo-load vector p_i^s and the semi-analytic method yields displacement derivatives which are scaled with respect to the analytic derivatives. In this case the accuracy of the derivative is only dependent on the accuracy of $\Delta f/\Delta b_i$.

If, as is often the case, $K(b_i)$ can not be factored as described above then the errors in the displacement derivatives may have a significantly different form. In this case the approximate pseudo-load vector is not a simple scaling of the true pseudo-load vector. Geometrically, this means that both the shape and length of the approximate pseudo-load vector may be incorrect. Also, since $\tilde{p}_i^s$ is a function of both the error matrix E_i and the displacement field, u, the accuracy of the derivatives may depend on the number of elements in the structural model and the location, within the model, of the element(s) dependent on b_i.

Derivative given by $\dfrac{\Delta u}{\Delta b_i} = -K^{-1}\tilde{p}_i^s$ where $p_i^s = \dfrac{\Delta K}{\Delta b_i}u$

- **K factorable** in b_i : $K = \bar{K} + f(b_i)K_{b_i}$

$$\tilde{p}_i^s = \frac{\Delta f(b_i)}{\Delta b_i}K_{b_i}u = (1 + \varepsilon_i)\frac{\partial f(b_i)}{\partial b_i}K_{b_i}u = (1+ \varepsilon_i)p_i^s$$

> Thus the derivatives **scale**: $\dfrac{\Delta u}{\Delta b_i} = (1+ \varepsilon_i)\dfrac{\partial u}{\partial b_i}$

- **K not factorable** in b_i : $K = \bar{K} + K(b_i)$

$$\tilde{p}_i^s = \frac{\Delta K(b_i)}{\Delta b_i}u = \left(\frac{\partial K(b_i)}{\partial b_i} + E_i\right)u = p_i^s + E_i u$$

> Derivatives **do not scale**: $\dfrac{\Delta u}{\Delta b_i} = \dfrac{\partial u}{\partial b_i} - K^{-1}(E_i u)$

Figure 2

EXAMPLE - CANTILEVERED BEAM

To illustrate these ideas, consider a cantilevered beam modeled as an assemblage of beam type finite elements. The element level stiffness matrix for the n-th element is shown in figure 3. Clearly, K_e (and thus K) is factorable for the element height (h) and width (w) variables but is not factorable for the length variable (l). Quantitatively then (based on our previous arguments) we expect the following when using the semi-analytic method: since w appears linearly, the derivatives of the displacements with respect to w will be exact and the relative error in the derivatives of the displacements with respect to h will be uniform and depend only on the accuracy of $\Delta h^3 / \Delta h$. However, the relative error in the displacement derivatives with respect to l may be non-uniform and may depend on the number of elements used to model the beam as well as on the accuracy of $\Delta K / \Delta l$. To confirm this, we will now derive analytical expressions for the relative error in these displacement derivatives.

Consider the finite element formulation for a beam element:

$$
K^e_n = \frac{2 E I_n}{l_n^3}
\begin{bmatrix}
6 & -3\,l_n & -6 & -3\,l_n \\
\vdots & 2\,l_n^2 & 3\,l_n & l_n^2 \\
\vdots & \cdot\cdot & 6 & 3\,l_n \\
sym & \dots & \dots & 2\,l_n^2
\end{bmatrix}
$$

$$
\text{where:} \quad I_n = \frac{w_n h_n^3}{12}
$$

> Clearly, K^e_n may be factored for the sizing variables h and w but not for the shape variable l

Figure 3

ERROR ANALYSIS FOR THE CANTILEVERED BEAM

   Consider the initially uniform cantilevered beam of length L shown in figure 4. The beam has a rectangular cross section and is subject to concentrated force (F) and moment (M) loadings at the tip. In order to investigate the accuracy of the tip displacement derivatives we can derive analytical expressions for the semi-analytic derivatives and compare them to the known true derivatives (Ref. 9). We begin with the expressions for the displacements and rotations along the length of the beam as shown below. Now let the beam be composed of N elements of length $l=L/N$ numbered from 1 to N, starting at the root. If the nodes are numbered from 0 to N, starting at the root, then the n-th node is located an $x=nl$. Substituting for x in the equations for $\mathbf{u}$ and θ yields a set of discretized equations for $\mathbf{u}$ and θ. To complete the derivation we need expressions for the entries of $\mathbf{K}^{-1}$ associated with the tip displacement d.o.f. and for the derivatives of the stiffness matrix with respect to the design variables b_n. The stiffness matrix derivatives are easily derived from the expression for the element level stiffness matrix shown previously. The necessary entries of $\mathbf{K}^{-1}$ can be obtained by differentiating the displacement vector with respect to the applied force (F). Substitution of these expressions into the equation for the tip displacement derivative will yield the desired analytical expression for the semi-analytic derivative.

We can determine analytical expressions for the errors introduced by finite differencing in the S-A method by using the exact beam element formulation.
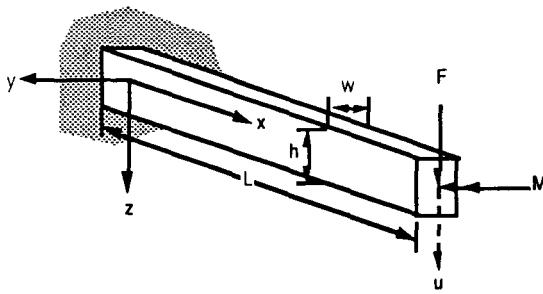
$$u(x) = \frac{(FL-M)x^2}{2EI} - \frac{Fx^3}{6EI}$$

$$\theta(x) = \frac{Fx^2}{2EI} + \frac{(M-FL)x}{EI}$$

Discretizing for a beam of N elements we can compute:

$$\frac{\Delta u^t}{\Delta b_n} = -\left\{\mathbf{K}^{-1}\right\}^T \frac{\Delta \mathbf{K}}{\Delta b_n}\mathbf{u} \quad \text{where} \quad \left\{\mathbf{K}^{-1}\right\}^T = \frac{\partial \mathbf{u}}{\partial F}$$

Figure 4

In figure 5 the expressions for the semi-analytic tip displacement derivatives with respect to the element heights and lengths (in terms of the exact derivatives) are shown. The expressions have been simplified to the case where $F=0$. For h, the stiffness matrix is factorable and, as was predicted, the relative error (ε) depends only on the finite difference parameter (c) and is the same for all elements making up the beam. For the element lengths, K is not factorable and the relative error is non-uniform. In this case the relative error depends not only on c, but also on the element number (n) and the number of elements (N) used to model the beam. Increasing either n or N will cause the relative error to become larger, while decreasing the value of c will give better accuracy. In Refs. 7 and 8 the tip displacement derivatives with respect to L ($\Delta u^t / \Delta L$) are investigated. This quantity is based on perturbations of all elements in the beam such that the quantity ΔL is distributed evenly among all elements. In this case the relative error in $\Delta u^t / \Delta L$ is equivalent to the average error (ε_{avg}) in $\Delta u^t / \Delta l_n$. For small values of c, ε_{avg} is approximately proportional to cN^2. Note that, as would be expected, in all cases the relative error approaches zero as c approaches zero.

- **Factorable** sizing variable h : $K = \overline{K} + f(h^3)K_h$

  Relative error is a function of c only: $\dfrac{\Delta u^t}{\Delta h} = (1 + \dfrac{c^2 + 3c}{3})\dfrac{\partial u^t}{\partial h}$

- **Non-factorable** shape variable l : $K = \overline{K} + K(l_n)$

  Relative error is a function of c, n, and N :

  $$\frac{\Delta u^t}{\Delta l_n} = \frac{\partial u^t}{\partial l_n}\left[\frac{2N - c^2(2n - 2N - 1) + c(12n^3 - 24\,n^2N - 18\,n^2 + 24nN + 8n - 2N + 1)}{2N(c+1)^3}\right]$$

  $$= \frac{\partial u^t}{\partial l_n}(1 + (\varepsilon(c, n, N)))$$

  Average error is a function of c, and N :

  $$\varepsilon_{avg} = \frac{1}{N}\sum_{n=1}^{N}\varepsilon(c, n, N) = \frac{1 - c(\frac{5}{2}N^2 - 2 - 1/N)}{(c+1)^3} - 1$$

Figure 5

COMPUTATIONAL RESULTS: SHAPE VARIABLES

To demonstrate the analytical results presented previously the cantilevered beam was modeled with beam elements and the tip displacement derivatives with respect to the element lengths were calculated numerically via the semi-analytic method. In the figure below these derivatives (for $c=.01$) are plotted (normalized by the true derivative) versus the element number for four different beam discretizations (5,10,15 and 20 elements). The numerically generated data points are represented by the symbols shown on the plot. The analytic results appear as the underlying curves. Note that the computed values are in complete agreement with the analytic values. As predicted, the relative error depends on both n and N. For this problem, the error increases with N and increases as we move along the beam from the root to the tip.

In figure 6 we also show the **average** error of the tip displacement derivatives, with respect to the element length, as a function of the number of elements in the beam. In this case the computed and predicted values are plotted for various values of c. In addition, the equivalent numerical data from Ref. 7 (represented by the square symbols) is also shown. Note, again, the excellent agreement between the computed and predicted errors. Clearly, the error decreases rapidly as c is decreased. For $c=.00001$ the average error is less than 1.0%. for $N=20$.
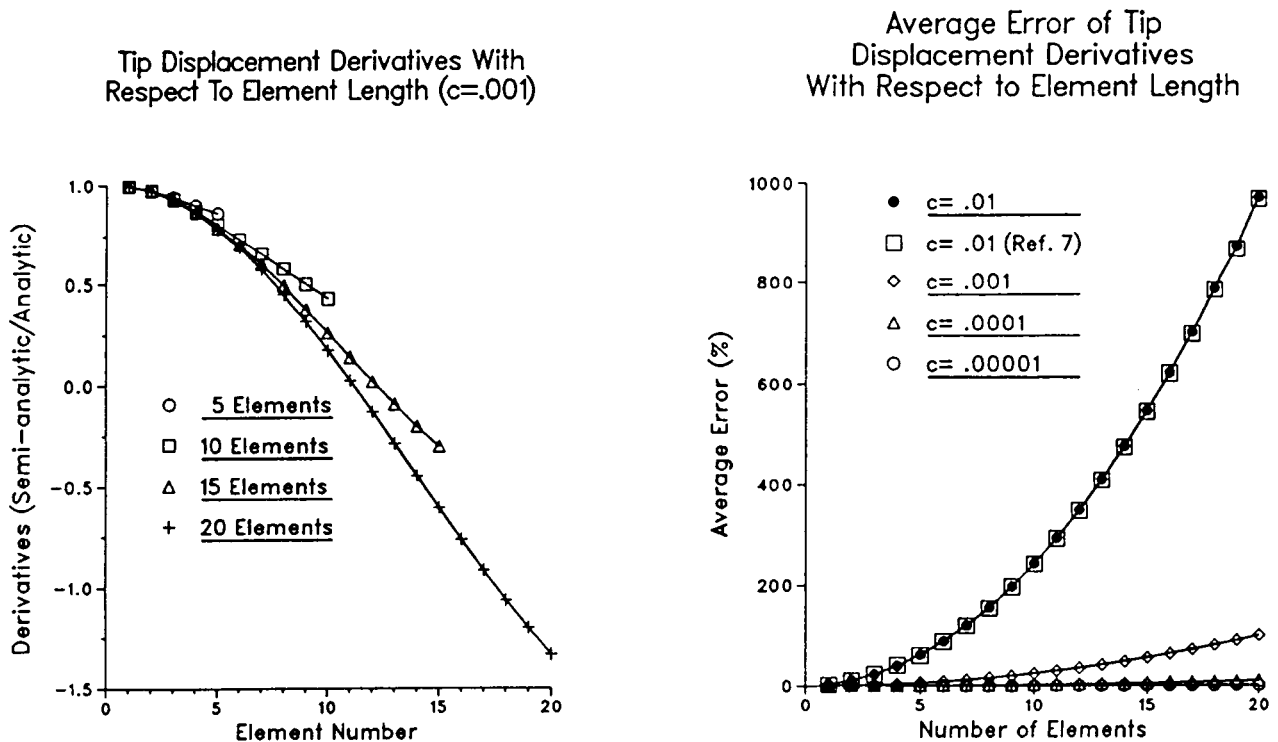

Figure 6

We have seen that for the cantilevered beam problem uniform errors occur for the tip displacement derivatives with respect to sizing design variables and nonuniform errors occur for shape variables. However, our general error analysis predicts that the nature of the error depends on the factorability of K and not necessarily on the classification of the design variable. We will now show that nonuniform derivative errors can (and usually do) occur for sizing design variables.

Consider the case of a rectangular cantilevered beam where the principal axes of the beam elements are oriented at some angle to the global coordinate system. In this case the stiffness matrix is not generally factorable for w and h since both variables contribute to the stiffnesses in the global coordinate system. As a result nonuniform error in the global displacement derivatives may occur. To investigate this analytically, consider the equivalent system shown in figure 7 where we want to calculate the derivative of the displacement at some angle θ to the beam's principal axes $(\Delta d^t/\Delta b_n)$. Using our previous analysis for displacement derivatives in the principal coordinate system we can develop the expression shown below for the relative error (ε_n) associated with $\Delta d^t/\Delta b_n$ (Ref.9). As shown, ε_n is a function of the relative errors associated with the principal displacement derivatives Generally, they combine such that ε_n will differ for each element. Under certain conditions, however, uniform errors will occur. Clearly, this will be the case when θ is some multiple of $\pi/2$. Uniform error will also occur when the loading in y and z directions are related by a scaling factor since the resulting displacements and displacement derivatives will also be simply scaled.
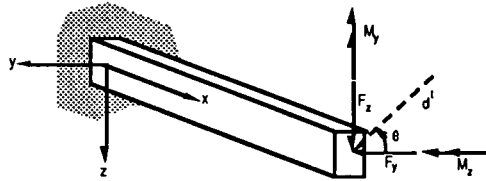
$$d^t = u^t \cos \theta + v^t \sin \theta$$

For sizing variables:

$$\frac{\Delta u^t}{\Delta b_n} = (1+ \varepsilon^u)\frac{\partial u^t}{\partial b_n}$$

$$\frac{\Delta v^t}{\Delta b_n} = (1+ \varepsilon^v)\frac{\partial v^t}{\partial b_n}$$

The relative error of the displacement derivative is given by :

$$\varepsilon_n = \frac{\dfrac{\Delta d^t}{\Delta b_n} - \dfrac{\partial d^t}{\partial b_n}}{\dfrac{\partial d^t}{\partial b_n}} = \frac{\varepsilon^u \dfrac{\partial u^t}{\partial b_n}\cos \theta + \varepsilon^v \dfrac{\partial v^t}{\partial b_n}\sin \theta}{\dfrac{\partial u^t}{\partial b_n}\cos \theta + \dfrac{\partial v^t}{\partial b_n}\sin \theta}$$

- **Uniform error** when $\theta = i\pi/2$ or $P_z = \gamma P_y$

- **Non-uniform error** otherwise

Figure 7

NONUNIFORM ERRORS - SIZING VARIABLES

To numerically illustrate that nonuniform errors can occur for displacement derivatives with respect to sizing design variables we calculated the derivative of the tip displacement for a cantilevered beam with respect to the element heights using the semi-analytic method. The beam was rotated so that its principal axes were oriented at 45 degrees relative to the global coordinate system. A tip force and moment were applied parallel to the global axes. In figure 8 the derivatives are plotted (normalized by the analytic derivatives) as a function of the element number for $c=0.01$ and various beam discretizations (5,10,15 and 20 elements). As expected, the derivatives depend on the element number and the number of element used to model the beam. In this case the relative error decreases with increasing N and decreases as we move along the beam from the root to the tip.

In figure 8 we also show the normalized tip displacement derivatives plotted versus the element number for $N=20$ and various values of c. Note that the relative error decreases rapidly as c is decreased. Also, the magnitudes of the errors are significantly less, for a given value of c, than we found for the derivatives with respect to the element lengths. For $N=20$, acceptable accuracy is obtained for values of c as large as 0.01.
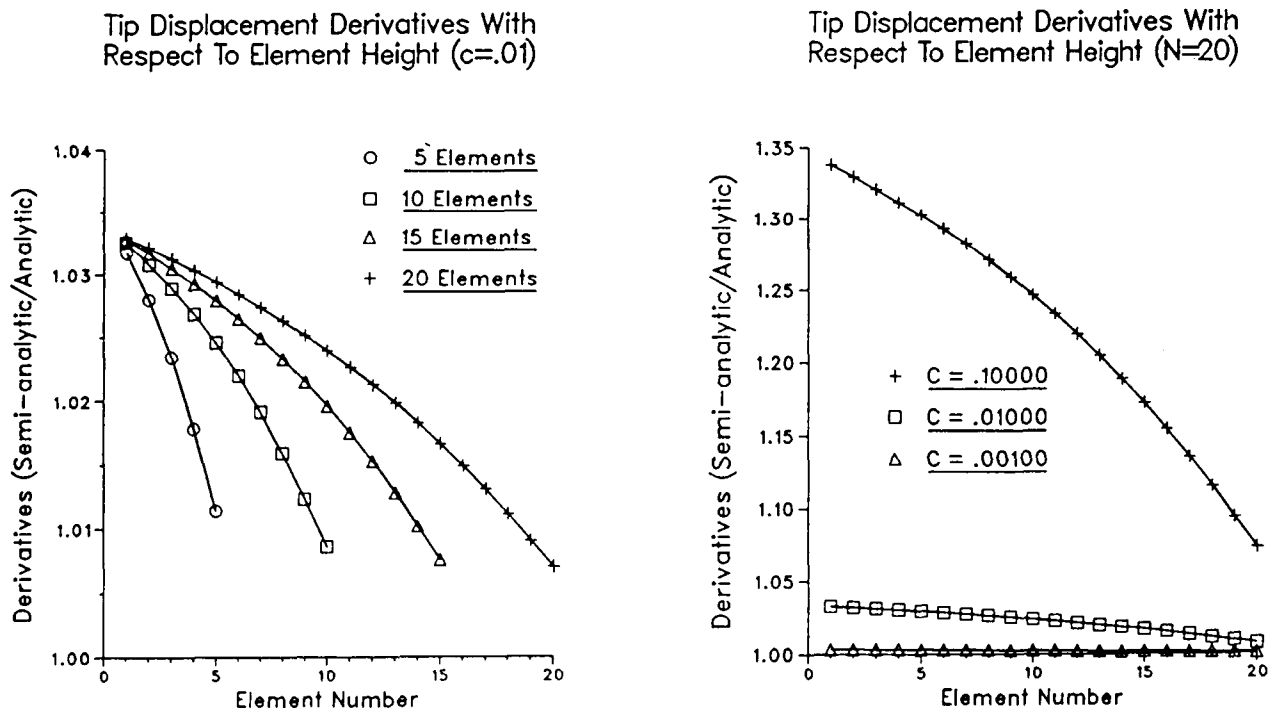

Figure 8

To study the accuracy of the semi-analytic method for more practical problems, consider the half model of an idealized automobile frame structure shown in figure 9. The structural model consists of 33 three-dimensional beam-type finite elements each having a rectangular cross section. The structure is simply supported at the front suspension attachment points (A) and loaded in the vertical direction at the rear suspension attachment points (B). Boundary conditions are applied to the center line grid points to enforce an anti-symmetric structural response. The net effect of the loading and boundary conditions is to cause torsion of the structure about the centerline. In this case we calculated the semi-analytic derivatives of the vertical displacement at point C with respect to the thickness, width, height and length of each element in the structure and compared them against the analytic derivatives. In all cases the accuracy of the derivatives varies from element to element. The results of the comparison are summarized in the plot below. For each type of design variable (length, height, width and thickness) the minimum, maximum and average errors in semi-analytic derivatives are plotted as a function of the finite difference parameter (c). Note that each type of design variable exhibits a different level of accuracy, for a given value of c, with the length variable being the worst and thickness being the best. This can be attributed to the varying degrees of nonlinearity of the stiffness matrix with respect to these variables. For a thin walled box beam the section properties are nearly linear functions of t and therefore the accuracy of the displacement derivatives with respect to t is much better than that for b, h and l. In general, the careful selection of design variables or other intermediate variables (e.g., beam section properties) for the derivative calculations will yield more accurate derivatives for any given value of c.
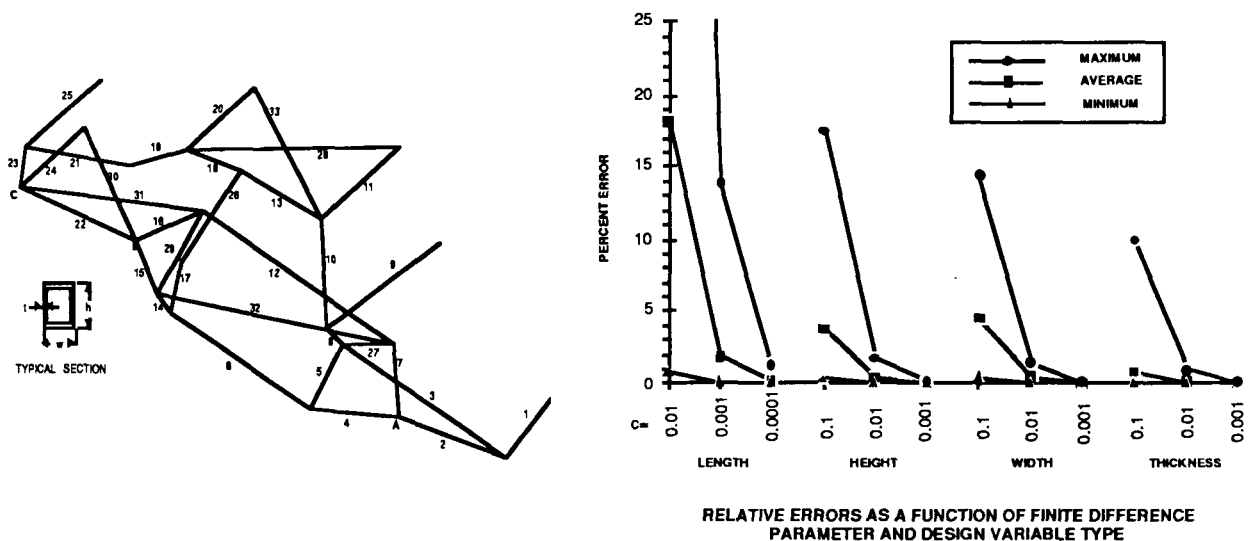

RELATIVE ERRORS AS A FUNCTION OF FINITE DIFFERENCE PARAMETER AND DESIGN VARIABLE TYPE

Figure 9

SUMMARY

The inaccuracy of the semi-analytic method for computing static displacement derivatives for both shape and sizing design variables has been shown to be the result of errors in the pseudo load vectors. Two types of errors were identified. In the first case the errors in the finite difference approximation to the stiffness matrix derivatives resulted in a scaling of the pseudo load vector which, in turn, causes the derivatives to be uniformly scaled relative to their true values. In this case the magnitude of the error depends only on the finite difference parameter, c. In the second case, errors in the finite difference operation lead to a distortion of the pseudo load vectors and nonuniform errors in the displacement derivatives. These errors may be dependent on the location (within the structure) of the element(s) associated with the design variable and the discretization of the structure, as well as c.

The results of the error analysis were demonstrated numerically for a cantilevered beam and an idealized automobile frame structure. It was observed that for a given value of c that the errors in the derivatives for shape design variables were significantly larger than those for sizing variables. However, in both cases the relative errors could be adequately controlled through the proper choice of the finite difference parameter. It should be noted that relatively small values for c may be required to compute sufficiently accurate derivatives. This suggests that it may be necessary to compute the finite difference approximations to the stiffness matrix derivatives in double precision to avoid roundoff errors. Also, by carefully choosing intermediate variables which appear linearly (or nearly so) in the stiffness matrix, greater accuracy in the the finite difference approximation can be obtained.

- Errors reported for the semi-analytic method have been shown to be due to errors in the finite difference approximation of the stiffness matrix derivatives
  - We can adequately control errors by careful choice of the finite difference parameter
- Errors may occur for both shape and sizing variables
  - For a given value of the finite difference parameter, errors in the derivatives of the shape variables were larger than the sizing variables

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Figure 10

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